Contributions to Discrete Mathematics

Volume 2, Number 1, Pages 61–66
ISSN 1715-0868

COLORING EDGES AND VERTICES OF GRAPHS WITHOUT SHORT OR LONG CYCLES

MARCIN KAMIŃSKI AND VADIM LOZIN

Abstract. Vertex and edge colorability are two graph problems that are NP-hard in general. We show that both problems remain difficult even for graphs without short cycles, *i.e.*, without cycles of length at most g for any particular value of g. On the contrary, for graphs without long cycles, both problems are shown to be solvable in poynomial time.

1. Introduction

All graphs in this paper are finite, undirected, without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$, the edge set by $E(G)$ and the *neighborhood* of a vertex $v \in V(G)$ (i.e., the set of vertices adjacent to v) by $N(v)$. The *degree* of v is $|N(v)|$. A graph every vertex of which has degree d is called *d-regular*. In particular, 3-regular graphs are called *cubic*. For notation not defined here we refer the reader to [4].

A *vertex coloring* of a graph G is an assignment of colors to its vertices such that any two adjacent vertices receive different colors. The *vertex colorability* problem is that of finding a vertex coloring of G that uses minimum number of different colors. This number is called the *chromatic number* of G. In its decision version, the problem asks to determine whether G admits a vertex coloring with at most k colors, where k is a constant. We shall refer to the decision version of the problem as *vertex k-colorability*.

From an algorithmic point of view, vertex colorability is a difficult problem, *i.e.*, it is NP-complete. Moreover, the problem remains difficult even under substantial restrictions, for instance, for graphs of vertex degree at most d (for each $d \geq 4$) or for line graphs. When restricted to line graphs, vertex colorability coincides with edge colorability of general graphs, i.e., the problem of finding an assignment of colors to edges such that every two edges with a vertex in common receive different colors and the number of different colors used is minimum. Similarly, edge k-colorability is the problem of determining whether an input graph admits an edge coloring with at most k colors. Holyer proved in [7] that edge colorability is an NP-complete problem by showing that edge 3-colorability is NP-complete for cubic graphs.

Received by the editors May 2, 2006, and in revised form November 25, 2006.
2000 *Mathematics Subject Classification.* 05C85, 05C15.
Key words and phrases. vertex coloring, edge coloring, NP-hard, polynomial time.

With a closer look at this solution one can realize that Holyer did not use triangles in his construction. Therefore, edge colorability remains difficult even for *triangle-free* graphs. In the present paper, we strengthen this result by showing that the problem is NP-complete for graphs without cycles of length at most g, for any particular value of g. We also prove a similar result for vertex colorability, extending the partial information on this topic obtained by Maffray and Preissmann [11].

Having proved the NP-completeness of vertex colorability and edge colorability on graphs without short cycles, we then turn to graphs without long cycles and show that both problems have polynomial-time solutions for such graphs.

2. Graphs without short cycles

The minimum length of a cycle in a graph G is called the *girth* of G. Graphs of large girth have been the subject of intensive investigations with respect to various problems (see *e.g.* [3, 9, 12]). In this section, we study computational complexity of vertex and edge colorability on graphs of large girth, and show that both problems are NP-hard for such graphs.

Theorem 2.1. *For any natural $g \geq 3$, the edge 3-colorability problem is NP-hard in the class of cubic graphs of girth at least g.*

Proof. Let G be a cubic graph. To prove the theorem we will present a polynomial reduction from G to a cubic graph G' with girth at least g such that G is 3-edge-colorable if and only if G' is. Without loss of generality, we can restrict ourselves to even values of g, since graphs of girth at least $g+1$ constitute a subclass of graphs of girth at least g.

For the proof, we shall need a cubic 3-edge-colorable graph of girth at least g. The existence of such a graph follows from a result of Imrich, who proved in [8] that for any integer $d > 2$, there are infinitely many regular graphs F of degree d whose girth $g(F)$ satisfies the inequality

$$g(F) > \frac{c \log n(F)}{\log(d-1)} - 2,$$

where c is a constant and $n(F)$ is the number of vertices of F. Therefore, for any $g \geq 3$, there is a regular graph F of degree g and of girth at least g. By replacing each vertex of F with a cycle of length g (see Figure 1 for illustration in case of $g = 4$), we obtain a cubic graph H of girth at least g. If g is even, we need only 2 colors to color the edges of each inserted cycle. The third color can be used to color the remaining edges of the graph (*i.e.*, the original edges of G). Hence H is 3-edge-colorable and we can also assume it is connected.

Let ab be an arbitrary edge in H and H_{ab} the graph obtained from H by removing the edge ab. Observe that the distance between a and b in H_{ab} is at least $g-1$ and H_{ab} is connected. Connectedness follows from the fact that

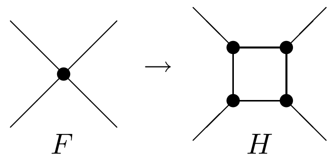

FIGURE 1. Replacement of a vertex of degree 4 with a cycle of length 4

every cubic 3-edge-colorable connected graph is bridgeless, *i.e.*, removing an edge does not disconnect the graph (see [4]).

Now we transform G into G' by replacing each of its edges with a copy of the graph H_{ab} as follows. Given an edge xy in G, we first delete this edge, then incorporate a copy of the graph H_{ab}, and finally, connect x to a and y to b. Since the distance between a and b in each copy of H_{ab} is at least $g-1$, the girth of G' is at least g.

Let us show that G is 3-edge-colorable if and only if G' is. It is not difficult to see that 3-edge-colorability of G together with 3-edge-colorability of H imply 3-edge-colorability of G'. To prove the converse statement, assume G' is 3-edge-colorable, and let x, y be a pair of vertices of G' that are adjacent in G. Also, let H_{ab}^{xy} be a copy of the graph H_{ab} such that x is adjacent to a and y is adjacent to b in G'. Our goal is to show that in any 3-edge-coloring of G' the edges xa and yb have the same color. Assume the contrary: the color of xa is 1, while the color of yb is 2. In the subgraph of G' induced by the edges of colors 1 and 2, every connected component is a cycle (since this subgraph is 2-regular). The edges xa and yb belong to a same component C of this subgraph, as these edges form a cutset in G'. Let P be the path connecting a to b in C and consisting of edges of the graph H_{ab}^{xy}. According to our assumption, the number of vertices in P is odd. But then the subgraph of H_{ab}^{xy} induced either by the edges of colors 1,3 or by the edges of colors 2,3 is not 2-regular. This contradiction shows that the edges xa and yb have the same color in any 3-edge-coloring of G', which completes the proof of the theorem. □

A direct consequence of the above theorem is the following corollary.

Corollary 2.2. *For any natural $g \geq 3$, the edge colorability problem is NP-hard in the class of graphs of girth at least g.*

In the rest of the section we study vertex colorability.

Theorem 2.3. *For every natural $k, g \geq 3$, vertex k-colorability is NP-hard in the class of graphs of girth at least g.*

Proof. The famous theorem of Erdős [5] states that for each pair of integers g, k ($g \geq 3$, $k \geq 2$) there exists a graph with girth at least g and chromatic number at least k. For a graph of girth at least g and chromatic number at least $k+1$, let H be its edge-minimal $(k+1)$-vertex-colorable subgraph.

By definition of H, for any edge ab, the graph H_{ab}, obtained from H by removing ab is k-vertex-colorable and vertices a, b receive the same color in every k-vertex-coloring of H_{ab}. Notice that the distance between a and b in H_{ab} is at least $g-1$.

To prove the theorem, we will show that an arbitrary graph G can be transformed in polynomial time into a graph G' of girth at least g such that G' is k-vertex-colorable if and only if G is. To this end, consider any short cycle C in G and any vertex v on C. Split the neighborhood of v into two parts A, B so that one of the neighbors of v on the cycle is in A, while the other one is in B. Remove vertex v from G and add a copy of the graph H_{ab} defined above along with the edges connecting a to the vertices of A and the edges connecting b to the vertices of B. It is easy to see that the graph obtained in this way is k-vertex-colorable if and only if G is.

Repeatedly applying this operation, we can destroy all short cycles in G, thus creating a graph which is k-vertex-colorable if and only if G is. Since the number of short cycles is bounded by a polynomial in the size of the input graph, the overall time required to destroy all cycles shorter than g is bounded by a polynomial. This proves the theorem. □

Corollary 2.4. *For any natural $g \geq 3$, the vertex colorability problem is NP-hard in the class of graphs with girth at least g.*

3. Graphs without long cycles

In this section we turn our attention to graphs without *long* cycles. Unlike graphs without short cycles, here we have to distinguish between graphs containing *no* long cycles and graphs without long *chordless* cycles. The maximum length of a cycle in a graph is called its *circumference*, while the *chordality* of a graph is the maximum length of a chordless cycle. Clearly, graphs of circumference at most c constitute a subclass of graphs of chordality at most c. If $c < 3$, these two classes are identical and coincide with the class of forests, *i.e.*, graphs every connected component of which is a tree. It is easy to see that restricted to trees both edge colorability and vertex colorability can be solved in polynomial time. A related result deals with the notion of partial k-trees, or equivalently, graphs of tree-width at most k. It has been shown in [1, 15] (resp. [14]) that edge colorability (resp. vertex colorability) of graphs of bounded tree-width is a polynomially solvable task. We use this result to show that both problems have polynomial solutions for graphs of bounded circumference.

Theorem 3.1. *For any natural c, there exists a constant k such that the tree-width of graphs of circumference at most c is at most k.*

Proof. If $c < 3$, then $k = 1$, as forests are exactly graphs of tree-width at most 1. For $c \geq 3$, we use the induction on c and the following two observations (the proof of which can be found, for instance, in [10]): first, the tree-width of a graph cannot be larger than the tree width of any of its blocks (maximal 2-connected subgraphs), and second, the addition of j vertices to a graph increases its tree-width by at most j.

Let G be a graph of circumference at most c and let H be a block in G. To prove the theorem, we will show that by deleting at most c vertices from H we can obtain a graph of circumference at most $c-1$. This is obvious in the case when H contains at most one cycle of length c. Now let C^1 and C^2 be two cycles of length c in H. Assume they are vertex disjoint. Consider two edges $e_1 \in C^1$ and $e_2 \in C^2$. Since H is 2-connected, there is a cycle in H containing both e_1 and e_2. In this cycle, one can distinguish two disjoint paths P' and P'', each of which contains the endpoints in C^1 and C^2, and the remaining vertices outside the cycles. The endpoints of the paths P' and P'' partition each of the cycles C^1 and C^2 into two parts. The larger parts in both cycles together with paths P' and P'' form a cycle of length at least $c+2$, contradicting the initial assumption. This contradiction shows that any two cycles of length c in H have a vertex in common. Therefore, removing the vertices of any cycle of length c from H results in a graph of circumference at most $c-1$, as required. □

Corollary 3.2. *For any natural c, the edge colorability and vertex colorability problems can be solved for graphs of circumference at most c in polynomial time.*

We complete the paper by discussing complexity of the problems on graphs of bounded chordality. For vertex colorability, such restrictions do not generally lead to an efficient solution: indeed, vertex 4-colorability remains NP-complete for graphs of chordality at most 12 (since it is NP-complete for graphs containing no path on 12 vertices as an induced subgraph [13]) and vertex 5-colorability remains NP-complete for graphs of chordality at most 8 (since it is NP-complete for graphs containing no path on 8 vertices as an induced subgraph [13]). However, for graphs of chordality at most 3 (chordal graphs) the problem of vertex colorability is known to be solvable in polynomial time (see [6]).

The authors are not aware of the status of the edge colorability problem on graphs of bounded chordality. However, polynomial-time solvability of its decision version can be easily derived from some known results.

Theorem 3.3. *For any natural k and c, the edge k-colorability problem on graphs of chordality at most c can be solved in polynomial time.*

Proof. Since graphs of maximum vertex degree $k+1$ are not k-edge-colorable, the problem can be restricted to graphs of degree at most k. It has been shown by Bodlaender and Thilikos [2] that if a graph has chordality at most c and maximum degree at most k, then its treewidth is at most $k\,(k-1)^{c-3}$. As we mentioned before, coloring the edges of graphs of bounded tree-width is a polynomially solvable task [15]. □

References

[1] H. L. Bodlaender, *Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees*, J. Algorithms **11** (1990), 631–643.

[2] H. L. Bodlaender and D. M. Thilikos, *Treewidth for graphs with small chordality*, Discrete Applied Mathematics **79** (1997), 45–61.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, *Acyclic colourings of planar graphs with large girth*, J. London Math. Soc. (2) **60** (1999), 344–352.

[4] R. Diestel, *Graph Theory*, Springer, 2000.

[5] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38.

[6] M. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, North Holland, 2004.

[7] I. Holyer, *The NP-completness of edge-coloring*, SIAM J. Comput. **10** (1981), 718–720.

[8] W. Imrich, *Explicit construction of regular graphs without small cycles*, Combinatorica **4** (1984), 53–59.

[9] X. Li and R. Luo, *Edge coloring of embedded graphs with large girth*, Graphs Combin. **19** (2003), 393–401.

[10] V. Lozin and D. Rautenbach, *On the band-, tree- and clique-width of graphs with bounded vertex degree*, SIAM J. Discrete Mathematics **18** (2004), 195–206.

[11] F. Maffray and M.Preissmann, *On the NP-completeness of the k-colorability problem for triangle-free graphs*, Discrete Mathematics **162** (1996), 313–317.

[12] O. J. Murphy, *Computing independent sets in graphs with large girth*, Discrete Appl. Math. **35** (1992), 167–170.

[13] J. Sgall and G.J. Wöginger, *The complexity of coloring graphs without long induced paths*, Acta Cybernet. **15** (2001), 107–117.

[14] J. A. Telle and A. Proskurowski, *Algorithms for vertex partitioning problems on partial k-trees*, SIAM J. Discrete Math. **10** (1997), 529–550.

[15] X. Zhou, S.-I. Nakano, and T. Nishizeki, *Edge-coloring partial k-trees*, J. Algorithms **21** (1996), 598–617.

RUTCOR - Rutgers University Center for Operations Research,
640 Bartholomew Road,
Piscataway, NJ, USA 08854.
E-mail address: mkaminski@rutcor.rutgers.edu

E-mail address: lozin@rutcor.rutgers.edu